UNIT-3 : MATHEMATICAL EXPECTATION

Define Mathematical expectation of a random variable and function of a Random variable. 10M

If x is a discrete random variable assuming values $x_1, x_2 \ldots \ldots x_n$ with probabilities $P_1, P_2, \ldots \ldots P_n$ respectively.
such that $\sum_{i=1}^{n} P_i = 1$ then the mathematical expectation of x is defined as $E(x) = \sum_{i=1}^{n} x_i P_i$. [Also, if the random variable x takes x_i with probability P_i. $(i = 1, 2, 3 \ldots \ldots)$

$$E(x) = \sum_{i=1}^{\infty} x_i P_i \quad \text{where } \sum_{i=1}^{\infty} P_i = 1$$

provided that the series is absolutely convergent.
i.e if $\sum_{i=1}^{\infty} |P_i x_i| = \sum_{i=1}^{\infty} P_i |x_i| < \infty$] it is not necessary for exam.

If x is a continuous random variable with probability density function $f(x)$, then its mathematical expectation is given by

$$E(x) = \int_{-\infty}^{\infty} x f(x) dx$$ [provided $\int_{-\infty}^{\infty} |x| f(x) dx < \infty$]

Hence ME of a random variable is nothing but its arthmetic mean.

Expectation of function of a random variable

If $g(x)$ is a function of a random variable then expected value of $g(x)$ is defined as

$$E(g(x)) = \sum g(x) p(x) \text{ for discrete}$$
$$= \int_{-\infty}^{\infty} g(x) f(x) dx \text{ for continuous}$$

[provided the above $\sum$ or $\int$ exist and is absolutely convergent.
i.e $\sum |g(x)| p(x) < \infty$
$\int |g(x)| f(x) dx < \infty$]

define moments, variance and covariance using ME

Moments :

Non central moments

if $g(x) = x^r$ then

$E(x^r) = \sum x^r p(x)$ for discrete

$= \int_{-\infty}^{\infty} x^r f(x) dx$ for continuous

this is r^{th} moment about origin (non-central moments)

if $r = 1$, $E(x) = \text{Mean} = \mu_1'$

if $r = 2$, $E(x^2) = \mu_2'$

$\mu_2 = \mu_2' - \mu_1'^2$

variance $= E(x^2) - [E(x)]^2$

this is variance of x

central moments:- if $g(x) = (x - \text{mean})^r$

$g(x) = [x - E(x)]^r$

$= (x - \bar{x})^r$ then

$E(x - \bar{x})^r = \sum (x - \bar{x})^r p(x)$ for discrete

$= \int_{-\infty}^{\infty} (x - \bar{x})^r f(x) dx$ for continuous

this is r^{th} moment about mean or central moments

if $r = 2$ $E(x - \bar{x})^2 = \mu_2 =$ variance

$E(x - \bar{x})^3 = \mu_3$

Statistical averages :-

using expectations,

i) Mean $= \mu_1' = E(x)$

ii) variance $= \mu_2 = E[x - E(x)]^2$

$= E(x^2) - [E(x)]^2$

iii) r^{th} moment about origin or non-central moments

$\mu_r' = E(x^r)$

iv) Central moments (or) r^{th} moment about mean

$\mu_r = E[(x - E(x))^r]$

Variance :- When deviation is taken from the mean ($\bar{x}$), the mean square deviation is called the variance of x and it is denoted by

$$\sigma_x^2 = \frac{1}{N}\sum f_i (x_i - \bar{x})^2$$

$$= \sum \left(\frac{f_i}{N}\right)(x_i - \bar{x})^2$$

$$= \sum p_i (x_i - \bar{x})^2$$

$$V(x) = \sigma_x^2 = E(x-\bar{x})^2$$

$$= E(x - E(x))^2$$

$$= \mu_2$$

$$= \sum (x-\bar{x})^2 p(x) \text{ for discrete}$$

$$= \int_{-\infty}^{\infty} (x-\bar{x})^2 f(x)\,dx \text{ for continuous}$$

$$V(x) = E(x - E(x))^2$$

$$= E(x^2) - (E(x))^2$$

$$\mu_2 = \mu_2' - \mu_1'^2$$

Covariance using Expectations:-

If x and y are two random variables then covariance between the variables x and y is defined as

$$\text{cov}(x,y) = E[(x - E(x))(y - E(y))]$$

$$= E[xy - yE(x) - xE(y) + E(x)E(y)]$$

$$= E(xy) - E(y)E(x) - E(x)E(y) + E(x)E(y)$$

$$\text{cov}(x,y) = E(xy) - E(x)E(y)$$

Independence of random variables using Expectation:

The random variables x and y are said to be independent if $E(xy) = E(x)E(y)$

i.e. $\text{cov}(x,y) = E(xy) - E(x)E(y)$

$$= E(x)E(y) - E(x)E(y)$$

$$\text{cov}(x,y) = 0$$

Laws of Expectations:-

State and prove addition theorem of expectation for two random variables.

(or)

Show that mathematical expectation of the sum of two random variables is the sum of their individual expectations.

For n Random variables,

$$E(x_1+x_2+\cdots+x_n) = E(x_1)+E(x_2)+\cdots+E(x_n)$$

Statement: The mathematical expectation of the sum of two random variables (n) is the sum of their individual expectations. Symbolically, if x and y are two random variables, then $E(x+y) = E(x)+E(y)$

Proof: Let x and y be two discrete random variables. Let x take the values $x_1, x_2 \ldots x_n$ with probabilities $p_1, p_2 \ldots p_i \ldots p_n$

(Where $p_i = P(x=x_i)$ and $\sum_i p_i = 1$

Let y take the values $y_1, y_2 \ldots y_j \ldots y_m$ with probabilities $p_1' \; p_2' \ldots p_m'$ Where $p_j' = P(y=y_j)$ and $\sum_j p_j' = 1$)

By definition, we have

$$E(x) = \sum_{i=1}^{n} x_i p_i$$

$$E(y) = \sum_{j=1}^{m} y_j p_j'$$

Let $P_{ij} = P(x=x_i, y=y_j)$

$$= P(x=x_i \cap y=y_j)$$

Since x and y are random variables their sum $(x+y)$ is also a random variable

$(x+y)$ take the values (x_i+y_j), $i = 1, 2, \ldots n$

$j = 1, 2, \ldots m$

$$\therefore E(x+y) = \sum_i \sum_j (x_i+y_j) P_{ij}$$

$$= \sum_i \sum_j x_i P_{ij} + \sum_i \sum_j y_j P_{ij}$$

$$= \sum_i x_i \sum_j P_{ij} + \sum_j y_j \sum_i P_{ij} \longrightarrow ①$$

$\sum_{j=1}^{m} P_{ij} = P_i$ (Marginal distribution of x) $i = 1, 2, \ldots$

Similarly $\sum_{i=1}^{n} P_{ij} = P'_j$ (Marginal distribution of y) $j = 1, 2, \ldots m$

Substituting for $\sum_j P_{ij}$ & $\sum_i P_{ij}$ in eq ①

We get $E(x+y) = \sum_i x_i P_i + \sum_j y_j P'_j$

$$E(x+y) = E(x) + E(y)$$

The result can be generalized to any number of random variables.

For 3 random variables x_1, x_2, x_3 we have

$$E(x_1 + x_2 + x_3) = E(x_1 + y) \quad \text{where } y = x_2 + x_3$$
$$= E(x_1) + E(y)$$
$$= E(x_1) + E(x_2 + x_3)$$
$$= E(x_1) + E(x_2) + E(x_3)$$

Similarly the theorem is true for n random variables.

$$E(x_1 + x_2 + \cdots + x_n) = E(x_1) + E(x_2) + \cdots + E(x_n)$$

Generalization of Addition theorem of Expectations

Statement: The mathematical expectation of sum of n random variables is equal to the sum of their expectations, provided all the expectations exist.

i.e if $x_1, x_2, \ldots x_n$ are n random variables, then $E(x_1 + x_2 + \cdots + x_n) = E(x_1) + E(x_2) + \cdots + E(x_n)$

i.e $E\left(\sum_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} E(x_i)$

Proof: For two random variables x_1 and x_2

We have $E(x_1 + x_2) = E(x_1) + E(x_2) \longrightarrow$ ①

∴ The theorem is true for $n=2$

Let us consider the theorem is true for $n=r$

$$E\left[\sum_{i=1}^{r} x_i\right] = \sum_{i=1}^{r} E(x_i) \longrightarrow ②$$

consider for $n = r+1$

$$E\left[\sum_{i=1}^{r+1} x_i\right] = E\left[\sum_{i=1}^{r} x_i + x_{r+1}\right]$$

$= E\left(\sum_{i=1}^{r} x_i\right) + E(x_{r+1})$ [$\because$ from ①]

$= \sum_{i=1}^{r} E(x_i) + E(x_{r+1})$ [$\because$ from ②]

$\Rightarrow \sum_{i=1}^{r+1} E(x_i)$

$\therefore E\left[\sum_{i=1}^{r+1} x_i\right] = \sum_{i=1}^{r+1} E(x_i)$

$\therefore$ the theorem is true for $n = r+1$

Hence by mathematical induction the theorem is true for all values of n.

$E(x_1 + x_2 + \cdots + x_n) = E(x_1) + E(x_2) + \cdots + E(x_n)$

Addition theorem of expectation for continuous random variables :-

statement :- If $x, y, z \ldots T$ are n random variables then $E(x+y+z+\cdots+T) = E(x) + E(y) + E(z) + \cdots + E(T)$

proof :-

Let x and y be continuous random variables with join p.d.f $f(x,y)$ and marginal pdf's $f(x)$ and $f(y)$ respectively.

By the definition of expectations

$$E(x) = \int_{-\infty}^{\infty} x f(x) dx$$

$$E(y) = \int_{-\infty}^{\infty} y f(y) dy$$

Now consider $E(x+y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y) f(x,y) dx\, dy$

$$E(x+y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f(x,y) dx\, dy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f(x,y) dx\, dy$$

$$= \int_{-\infty}^{\infty} x \left\{\int_{-\infty}^{\infty} f(x,y) dy\right\} dx + \int_{-\infty}^{\infty} y \left\{\int_{-\infty}^{\infty} f(x,y) dx\right\} dy$$

$$= \int_{-\infty}^{\infty} x f(x) dx + \int_{-\infty}^{\infty} y f(y) dy$$

$$E(x+y) = E(x) + E(y)$$

this result can be generalize in the case of 3 random variables $E(x+y+z) = E(x+y) + E(z)$

$= E(x) + E(y) + E(z)$

Hence we have in general by mathematical induction,

$$E(x+y+z+\dots+T) = E(x)+E(y)+E(z)+\dots+E(T)$$

Multiplication theorem of Expectations

State and prove the multiplication theorem for two (n) independent random variables.

(or)

Show that the mathematical expectation of product of two (n) independent random variables is the product of their expectations.

Statement: the mathematical expectation of product of two independent random variables is equal to the product of their expectations.

Symbolically, if x and y are two independent random variables, then $E(xy) = E(x) \cdot E(y)$

Symbolically, if $x, y, z \dots T$ are n independent random variables, then $E(xyz\dots T) = E(x)E(y)E(z) \dots E(T)$

To prove the statement let us consider two independent random variables x and y.

Let x take the values $x_1, x_2, \dots x_i \dots x_n$ with probabilities $P_1, P_2 \dots P_i \dots P_n$ respectively, such that $\sum_{i=1}^{n} P_i = 1$

and y take the values $y_1, y_2, \dots y_j \dots y_m$ with probabilities $P_1', P_2' \dots P_j' \dots P_m'$ respectively, such that $\sum_{j=1}^{m} P_j' = 1$

By definition, we have $E(x) = \sum_i x_i P_i$ and $E(y) = \sum_j y_j P_j'$

Let $P_{ij} = P(x = x_i \cap y = y_j)$

$= P(x = x_i) \cdot P(y = y_j)$

$= P_i P_j'$

since x and y are random variables, their product (xy) is also a random variables.

$\therefore xy$ take the values $x_i y_j$ $i = 1, 2, \ldots n$ $j = 1, 2, \ldots m$

$$\therefore E(xy) = \sum_i \sum_j x_i y_j P_{ij} = \sum_i \sum_j x_i y_j P_i P_j' \quad (\because P_{ij} = P_i P_j')$$

$$= \sum_i x_i P_i \sum_j y_j P_j'$$

$$\boxed{E(xy) = E(x) E(y)}$$

this result can be generalize in the case of 3 independent random variables x, y, z

$$E(xyz) = E(xy) E(z)$$

$$\boxed{E(xyz) = E(x) E(y) E(z)}$$

Hence by mathematical induction we shall get for n independent random variables.

$$\boxed{E(xyz \ldots T) = E(x) E(y) E(z) \ldots E(T)}$$

Generalization of multiplication theorem of expectations:-

statement :- the mathematical expectation of the product of n independent random variables is equal to the product of their expectations.

i.e if $x_1, x_2 \ldots x_n$ are n independent random variables then symbolically $E(x_1 x_2 x_3 \ldots x_n) = E(x_1) E(x_2) \ldots E(x_n)$. or

$$E\left(\prod_{i=1}^{n} x_i\right) = \prod_{i=1}^{n} E(x_i)$$

proof : for two independent random variables x_1 and x_2

We have $E(x_1 * x_2) = E(x_1) E(x_2) \ldots (1)$

Let us suppose that the theorem is true for $n = r$

$$\therefore E\left(\prod_{i=1}^{r} x_i\right) = \prod_{i=1}^{r} E(x_i) \ldots (2)$$

consider for $n = r+1$

$$E\left(\prod_{i=1}^{r+1} x_i\right) = E\left[\left(\prod_{i=1}^{r} x_i\right) \cdot (x_{r+1})\right]$$

$$= E\left(\prod_{i=1}^{r} x_i\right) \cdot E(x_{r+1}) \quad [\because \text{from } ①]$$

$$= \prod_{i=1}^{r} E(x_i) \cdot E(x_{r+1}) \quad [\because \text{from } ②]$$

$\therefore E\left(\prod_{i=1}^{r+1} x_i\right) = \prod_{i=1}^{r+1} E(x_i)$

$\therefore$ the theorem is true for $n = r+1$

Hence by mathematical induction, the theorem is true for all values of n.

$$E\left(\prod_{i=1}^{n} x_i\right) = \prod_{i=1}^{n} E(x_i)$$

Multiplication Theorem of ME for continuous random variable :-

Statement:- If $x, y, z \ldots T$ are n independent random variables then $E(xyz \ldots T) = E(x)E(y)E(z) \ldots E(T)$

proof:- If x is a continuous Random variable then its expectation can be defined as

$$E(x) = \int_x x f(x)\,dx$$

If y is a continuous Random variable then its expectation can be defined as

$$E(y) = \int_y y f(y)\,dy$$

If xy is a continuous Random variable then its expectation can be defined as

$$E(xy) = \int_x \int_y xy\, f(x,y)\,dx\,dy$$

In the statement x and y are independent Random variables then $f(x,y) = f(x).f(y)$

$$E(xy) = \int_x \int_y xy\, f(x) f(y)\,dx\,dy$$

$$= \int_x x f(x)\,dx \int_y y f(y)\,dy$$

$$E(xy) = E(x)\,E(y)$$

This result can be generalize to the case of more than two independent Random variables x, y, z

We have $E(xyz) = E(xy)\,E(z)$

$$E(xyz) = E(x)\,E(y) \ldots E(z)$$

Hence by mathematical induction We shall get for n independent Random variables.

$E(XYZ \cdots T) = E(X)E(Y)E(Z) \cdots E(T)$

Properties of Expectations :-

1) Show that $E(a) = a$ where a is a constant.

Proof: We have $E(a)$ = mean of a

$$= \frac{a+a+\cdots+a}{n}$$

$$= \frac{na}{n}$$

$$E(a) = a$$

* Show that $E(c) = c$

Proof: Let x be continuous random variable with p.d.f $f(x)$ $\quad E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$

$$E(c) = \int_{-\infty}^{\infty} c f(x)\,dx$$

$$= c\int_{-\infty}^{\infty} f(x)\,dx \qquad \left[\int_{-\infty}^{\infty} f(x)\,dx = \text{Total probability} = 1\right]$$

$$= c(1) = c$$

$$\therefore E(c) = c$$

2) Show that $E(ax) = aE(x)$

Proof: Let x be a continuous Random variable.

$$E(ax) = \int_{-\infty}^{\infty} ax f(x)\,dx \rightarrow E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= a\int_{-\infty}^{\infty} x f(x)\,dx$$

$$= a\,E(x)$$

3) If x and y are two random variables $f(x), f(y)$ are p.d.f of x and y

$$E(ax+by) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (ax+by) f(x,y)\,dx\,dy$$

where $f(x,y)$ is the joint p.d.f of x and y

$$E(ax+by) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ax f(x,y)\,dx\,dy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} by f(x,y)\,dx\,dy$$

$$= a\int_{-\infty}^{\infty} x\left\{\int_{-\infty}^{\infty} f(x,y)\,dy\right\}dx + b\int_{-\infty}^{\infty} y\left\{\int_{-\infty}^{\infty} f(x,y)\,dx\right\}dy$$

$$= a\int_{-\infty}^{\infty} x f(x)\,dx + b\int_{-\infty}^{\infty} y f(y)\,dy \qquad \left\{\because \int_{-\infty}^{\infty} f(x,y)\,dy = f(x),\ \int_{-\infty}^{\infty} f(x,y)\,dx = f(y)\right.$$

$$E(ax+by) = a\,E(x) + b\,E(y)$$

4) If x is a random variable and $\underline{a}$ is constant then show that

i) $E(a\psi(x)) = aE(\psi(x))$

ii) $E(\psi(x)+a) = E(\psi(x))+a$

where $\psi(x)$ is a function of x

Proof:

Let x be a continuous Random variable

By def $E(X) = \int x f(x)dx$

i) $E[a\psi(x)] = \int_{-\infty}^{\infty} a\,\psi(x) f(x)dx$

$= a\int_{-\infty}^{\infty} \psi(x) f(x)dx$

$= a\,E[\psi(x)]$

ii) $E[\psi(x)+a] = \int_{-\infty}^{\infty} (\psi(x)+a) f(x)dx$

$= \int_{-\infty}^{\infty} \psi(x) f(x)dx + a\int_{-\infty}^{\infty} f(x)dx$

$= E[\psi(x)]+a \qquad \left[\because \int_{-\infty}^{\infty} f(x)dx = 1\right]$

Note:

i) $E(X-\bar{X}) = 0$

ii) $E(4x+5) = 4E(x)+5$

iii) $E(X^2) \neq [E(x)]^2 \qquad \because E(x^2) = \int x^2 f(x)dx$, $(E(x))^2 = \left[\int x f(x)dx\right]^2$

iv) $E\left(\frac{1}{x}\right) \neq \frac{1}{E(X)}$

v) $E(\log x) \neq \log[E(x)]$

Imp

properties of variance :-

1) show that $var(c) = 0$, c is a constant.

Proof: $var(c) = E(c - E(c))^2$

$= E(c-c) \qquad [\because E(c) = c]$

$var(c) = 0$

2) Show that $var(ax) = a^2 v(x)$ where a is constant

Proof: $var(ax) = E[ax - E(ax)]^2 \qquad [\because v(x) = E(x$

$= E[ax - aE(x)]^2$

$= E[a(x-E(x))]^2$
$= a^2 E(x-E(x))^2$

$var(ax) = a^2 V(x)$

3) Show that $var(ax+b) = a^2 Var(x)$ where a, b are constants.

proof: $var(ax+b) = E[ax+b-E(ax+b)]^2$

$= E(ax+b-aE(x)-b)$ $\boxed{\because V(x) = E(x-E(x))^2}$

$= E(a(x-E(x)))^2$

$= a^2 E[x-E(x)]^2$

$V(ax+b) = a^2 V(x)$

4) Show that $V(ax+by) = a^2 V(x) + b^2 V(y) + 2ab\, cov(x,y)$ where a, b are constants.

proof: $V(ax+by) = E[ax+by-E(ax+by)]^2$

$= E[ax+by-aE(x)-bE(y)]^2$ $\boxed{\because V(x) = E[x-E(x)]^2}$

$= E[a(x-E(x)) + b(y-E(y))]^2$

$= a^2 E(x-E(x))^2 + b^2 E(y-E(y))^2 + 2ab\, E[(x-E(x))(y-E(y))]$

$V(ax+by) = a^2 V(x) + b^2 V(y) + 2ab\, cov(x,y)$

5) $V(ax-by) = a^2 V(x) + b^2 V(y) - 2ab\, cov(x,y)$

Results :- $V(ax+b) = a^2 V(x)$

i) If $b = 0 \Rightarrow V(ax) = a^2 V(x)$

ii) If $a = 0 \Rightarrow V(b) = 0$

iii) If $a = 1 \Rightarrow V(x+b) = V(x)$

CO-variance :-

If x and y are two Random variables with respective Expectative values $E(x)$ and $E(y)$ or means $[(\bar{x})$ and $(\bar{y})]$ covariance between them is defined as

$cov(x,y) = E[(x-E(x))(y-E(y))]$

$cov(x,y) = E[xy - E(y)x - yE(x) + E(x)E(y)]$

$cov(x,y) = E(xy) - E(x)E(y) - E(y)E(x) + E(x)E(y)$

$cov(x,y) = E(xy) - E(x)E(y)$

If x and y are independent random variables then $E(xy) = E(x)E(y)$

$cov(x,y) = E(x)E(y) - E(x)E(y)$

$= 0$

Properties of covariance:-

1) show that $cov(ax, by) = ab\, cov(x,y)$

Proof: $cov(ax, by) = E[(ax - E(ax))(by - E(by))]$

$cov(ax, by) = E[(ax - aE(x))(by - bE(y))]$

$cov(ax, by) = ab\, E[(x - E(x))(y - E(y))]$

$cov(ax, by) = ab\, cov(x,y)$

2) show that $cov(x+a, y+b) = cov(x,y)$

Proof: $cov(x+a, y+b) = E[(x+a - E(x+a))(y+b - E(y+b))]$

$= E[(x+\cancel{a} - E(x) - \cancel{a})(y+\cancel{b} - E(y) - \cancel{b})]$

$= E[(x - E(x))(y - E(y))]$

$= cov(x,y)$

3) show that $cov(ax+b, cy+d) = ac\, cov(x,y)$

Proof: $cov(ax+b, cy+d)$

$= E[(ax+b - E(ax+b))(cy+d - E(cy+d))]$

$= E[(ax+\cancel{b} - aE(x) - \cancel{b})(cy+\cancel{d} - cE(y) - \cancel{d})]$

$= ac\, E[(x - E(x))(y - E(y))]$

$cov(ax+b, cy+d) = ac\, cov(x,y)$

4) covariance is independent of change of origin but not scale.

$$cov\left(\frac{x-a}{h}, \frac{y-b}{k}\right) = hk\, cov(u,v)$$

$$u = \frac{x-a}{h}, \quad v = \frac{y-b}{k}$$

theorem: show that variance of a linear combination of Random variables.

statement: Let $x_1, x_2 \ldots x_n$ are n random variables and if $a_1, a_2, \ldots a_n$ are any n constants then

$$V\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i^2 V(x_i) + 2\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \operatorname{cov}(x_i, x_j)$$

proof:

Let $U = \sum_{i=1}^{n} a_i x_i$

$$U = a_1x_1 + a_2x_2 + \ldots + a_nx_n \longrightarrow ①$$

Taking expectation on both sides

$$E(U) = a_1E(x_1) + a_2E(x_2) + \ldots + a_nE(x_n) \longrightarrow ②$$

subtracting eq ② from eq ① we get

$$U - E(U) = a_1(x_1 - E(x_1)) + a_2(x_2 - E(x_2)) + \ldots + a_n(x_n - E(x_n)) \longrightarrow ③$$

squaring on both sides eq ③

$$[U - E(U)]^2 = a_1^2[x_1 - E(x_1)]^2 + a_2^2[x_2 - E(x_2)]^2 + \ldots + a_n^2[x_n - E(x_n)]^2 + 2\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j (x_i - E(x_i))(x_j - E(x_j))$$

Taking expectation on both sides

$$E[U - E(U)]^2 = a_1^2 E[x_1 - E(x_1)]^2 + a_2^2 E[x_2 - E(x_2)]^2 + \ldots + a_n^2 E[x_n - E(x_n)]^2 + 2\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j E[(x_i - E(x_i))(x_j - E(x_j))]$$

We know that $V(x) = E(x - E(x))^2$

$$V(U) = a_1^2 V(x_1) + a_2^2 V(x_2) + \ldots + a_n^2 V(x_n) + 2\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \operatorname{cov}(x_i, x_j)$$

Since $U = \sum_{i=1}^{n} a_i x_i$

$$V\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i^2 V(x_i) + 2\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \operatorname{cov}(x_i, x_j)$$

Result :- $v(a_1x_1 + a_2x_2 + \cdots + a_nx_n) = a_1^2 v(x_1) + a_2^2 v(x_2) + \cdots + a_n^2 v(x_n) + 2\sum_{i=1}^{n}\sum_{j=1}^{n} a_ia_j \operatorname{cov}(x_i, x_j)$

i) If $a_1 = a_2 = \cdots = a_n = 1$ then

$$v(x_1 + x_2 + \cdots + x_n) = v(x_1) + v(x_2) + \cdots + v(x_n) + 2\sum_{i=1}^{n}\sum_{j=1}^{n} \operatorname{cov}(x_i, x_j)$$

ii) If the random variable $x_1, x_2 \cdots x_n$ are independent then $\operatorname{cov}(x_i, x_j) = 0$

$$\therefore v(x_1 + x_2 + \cdots + x_n) = v(x_1) + v(x_2) + \cdots + v(x_n)$$

(Addition theorem of variance)

iii) If $a_1 = a_2 = 1$ and $a_3 = a_4 = \cdots = a_n = 0$ then

$$v(x_1 + x_2) = v(x_1) + v(x_2) + 2\operatorname{cov}(x_1, x_2)$$

If x_1, x_2 are independent then

$$v(x_1 + x_2) = v(x_1) + v(x_2) \qquad \operatorname{cov}(x_1, x_2) = 0$$

iv) If $a_1 = 1$ $a_2 = -1$ and $a_3 = a_4 = \cdots = a_n = 0$

then $v(x_1 - x_2) = v(x_1) + v(x_2) - 2\operatorname{cov}(x_1, x_2)$

If x_1, x_2 are independent then

$$v(x_1 - x_2) = v(x_1) + v(x_2)$$

$v(x_1 \pm x_2) = v(x_1) + v(x_2)$ if x_1, x_2 are independent.